Communications in Nonlinear Science and Numerical Simulation 114 (2022) 106452

Contents lists available at ScienceDirect

Communications in Nonlinear Science and Numerical Simulation

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Modeling chaotic systems: Dynamical equations vs machine learning approach

Tongfeng Weng [a,*], Huijie Yang [b], Jie Zhang [c], Michael Small [d,e,*]

[a] Institute of Information Economy and Alibaba Business College, Hangzhou Normal University, Hangzhou 311121, PR China
[b] Business School, University of Shanghai for Science and Technology, Shanghai 200093, PR China
[c] Institute of Science and Technology for Brain-Inspired Intelligence, Fudan University, Shanghai 200433, PR China
[d] Complex Systems Group, Department of Mathematics and Statistics, The University of Western Australia, Crawley, WA 6009, Australia
[e] Mineral Resources, CSIRO, Kensington, WA 6151, Australia

ARTICLE INFO

Article history:
Received 21 July 2020
Received in revised form 26 February 2022
Accepted 21 March 2022
Available online 20 June 2022

Keywords:
Reservoir computing approach
Chaotic systems
Recurrence
Synchronization

ABSTRACT

Chaotic systems are ubiquitous in the real world, but often analytical models remain inaccessible. We find that a machine-learning method known as "reservoir computing" provides an alternative feasible way for modeling chaotic systems rather than conventional dynamical equations. Specifically, we show that recurrence in temporal and spatial scales of the trained reservoir system are indistinguishable from that of an observed chaotic system. Furthermore, by sharing a common signal, dual synchronization between a chaotic system and its learned reservoir system can be achieved successfully. In the same manner, we show that the identical synchronization also emerges on their coupled system. These findings reveal that reservoir computing approach would be excellent candidate for modeling a great variety of chaotic systems.

1. Introduction

Since the seminal work of Lorenz [1], the concept of chaos has attracted considerable interest over many decades [2–5]. The importance of chaotic dynamics originates from its broad relevance in our life — ranging from the solar activity [4], weather forecasting [3], financial markets [6], to human health [7]. Meanwhile, a growing number of studies reveal that chaotic systems have a great variety of intriguing features, for example, self-similar structure [3] and unstable periodic orbits [8]. Generally, if one wishes to predict, classify or control a chaotic system, the most effective way is to model such system via the analytical equations. Unfortunately chaotic systems with extant dynamical equations are scarce.

Recently, significant advances have been achieved in model-free prediction of chaotic systems via methods of reservoir computing [9–15]. In particular, a substantial body of research demonstrated that this approach is capable of predicting low-dimensional chaotic systems [9], inferring unmeasured variables [16,17], detecting chaotic signals [18], and even forecasting large spatiotemporally chaotic systems [11]. Beyond prediction, there is a growing industry in exploring long-term ergodic behavior contained in the trained reservoir system, such as Lyapunov exponents [19], attractor reconstruction [20], and correlation dimension [21]. In this context, an intriguing problem arises: can we use reservoir computing approach to accurately describe chaotic systems instead of dynamical equations?

* Corresponding authors.
E-mail addresses: wtongfeng2006@163.com (T. Weng), michael.small@uwa.edu.au (M. Small).

https://doi.org/10.1016/j.cnsns.2022.106452
1007-5704/

In this paper, we consider the equivalence between a reservoir computer model and dynamical equations in the context of recurrence — a fundamental characteristic of a dynamical system [6,22]. Specifically, we show that a great variety of recurrence properties of a chaotic system remain constant when moving from its dynamical model to the reservoir computer system. We further show that by transforming both into networks, a number of network statistics are also the same. Remarkably, we find that synchronization can be achieved between a chaotic system and its learned reservoir system as well as in their coupled system. Our work reveals that rather than dynamical equations, reservoir computing approach provides a new path for accurately modeling chaotic systems.

2. Model description

2.1. Reservoir computing approach

We start with introducing the framework of the reservoir computing approach. Generally, its architecture contains three basic parts: an input layer coupled with an input vector $\mathbf{u}$, a reservoir network in the middle layer, and an output layer coupled with an output vector $\mathbf{v}$. Following Jaeger's design [9], we define the evolving equations of the reservoir vector $\mathbf{r}$ and the output vector $\mathbf{v}$ as follows:

$$\mathbf{r}(t+1) = (1-\alpha)\mathbf{r}(t) + \alpha \tanh\left(\mathbf{A}\mathbf{r}(t) + \mathbf{W}_{in}\begin{pmatrix} b_{in} \\ \mathbf{u}(t) \end{pmatrix}\right), \tag{1}$$

$$\mathbf{v}(t+1) = \mathbf{W}_{out}\begin{pmatrix} b_{out} \\ \mathbf{u}(t) \\ \mathbf{r}(t+1) \end{pmatrix}, \tag{2}$$

where $\mathbf{A}$, $\mathbf{W}_{in}$, α, b_{in}, and b_{out} are the reservoir parameters. They are usually given in advance before training. Here, we set these reservoir parameters according to the commonly used method [23]. In this context, the output weighted matrix $\mathbf{W}_{out}$ is the sole fitting parameter to be determined in the training phase. For a chaotic system of interest, this parameter can be in fact expressed in terms of the training data analytically [24,25]. After training phase, when $\mathbf{v}(t+1)$ is adopted as $\mathbf{u}(t+1)$, the reservoir computing system becomes a self-evolving dynamical system based on Eqs. (1) and (2). Next, we will explore how the trained reservoir computer related to its learned chaotic system.

2.2. Recurrence time of chaotic systems and their learned reservoir systems

We first apply the reservoir computing approach to learn the Hénon map given by:

$$x_{n+1} = 1 + y_n - 1.4x_n^2, \tag{3}$$

$$y_{n+1} = 0.3x_n. \tag{4}$$

We generate the 4×10^5 observations from this map and use the first 2600 points with $\alpha = 0.25$ and the input $\mathbf{u} = (x, y)$ for training. After the training stage, we consider a trajectory of length $N = 4 \times 10^5$ of this reservoir system and randomly choose a reference point X_0. For a given radius ϵ, we computer the set $S(\epsilon)$ of recurrence points such that $S(\epsilon) = \{X_{t_i} : \| X_{t_i} - X_0 \| < \epsilon\}$. Then, we extract the recurrence times from this recurrence set defined as $T = \{T(i) : T(i) = t_{i+1} - t_i\}$. Interestingly, when observing the mean recurrence time $\langle T \rangle$ and the variation σ of the recurrence times T, a clear scaling law emerges such that $\langle T \rangle \sim \epsilon^d$ and $\sigma \sim \epsilon^\gamma$, see Fig. 1(a) and (b). We find that these scaling behaviors match that expected of the Hénon map exactly. Meanwhile, the slopes estimated from these lines are $d = -1.22$ and $\gamma = -2.47$, which are consistent with the results reported in Refs. [22,26]. We further apply the same calculation to the Lorenz system ($dx/dt = 10(y-z)$, $dy/dt = -xz + 60x - y$, $dz/dt = xy - 8/3z$). Remarkably, equally convincing scaling behaviors are achieved between the Lorenz system and its reservoir computing model, as described in Fig. 1(c) and (d). Our findings reveal that the recurrence times of the reservoir computing system are indistinguishable from that of a chaotic system for modeling.

3. Simulation results

3.1. Recurrence networks of chaotic systems and their learned reservoir systems

Besides the recurrence times, we further explore their equivalence from complex network perspective for which a number of network metrics can be employed. In fact, network science has recently been widely employed to characterize chaotic systems of interest [6,7]. Here, we adopt a network transform based on the recurrence method [27–29]. Specifically, for a dynamical system given by $\{X_i\}_{i=1}^N$, the resulting network is

$$A_{ij}(\epsilon) = \Theta(\epsilon - \| X_i - X_j \|) - \delta_{ij}, \tag{5}$$

where $\Theta(\cdot)$ is the Heaviside function, ϵ is a threshold value, and δ_{ij} is the Kronecker delta. After constructing the transformed network, we observe its local, intermediate, and global network properties in terms of the mean degree

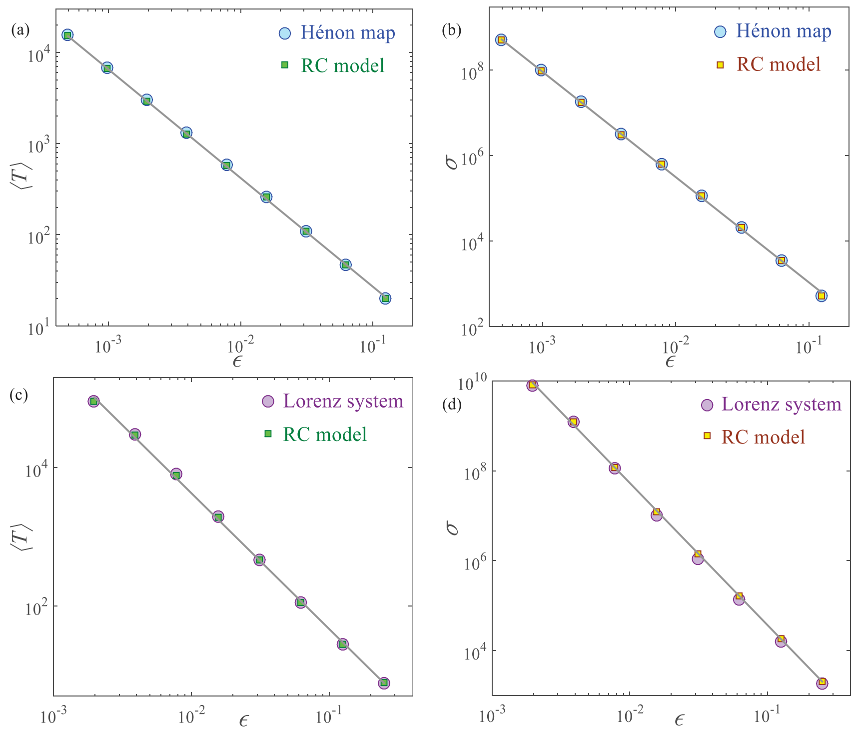

Fig. 1. Comparison of the Hénon map and its reservoir computing system with respect to (a) the mean recurrence time $\langle T\rangle$ and (b) the variation σ of recurrence times. (c) and (d) The similar comparison taking place on the Lorenz system.

$\langle K\rangle$, the clustering coefficient C, and the energy E [30], respectively. We then apply these network statistics to analyze the previous chaotic systems and their associated reservoir computing systems. Interestingly, we find that for either the Hénon map or the Lorenz system, the three network measurements of the reservoir computing system are almost the same as that of the original chaotic system, see Fig. 2. In particular, we show that the mean degree $\langle K\rangle$ and the clustering coefficient C monotonically increase with increasing the threshold value ϵ, while for the energy E, it increases on small threshold values and then gradually decreases. These findings reveal that the reservoir computing system is identical to a chaotic system of interest even from complex network perspective.

3.2. Synchronization between chaotic systems and reservoir computing systems

Moreover, we show that synchronization can occur between the dynamical equations and the reservoir computing representation of a chaotic system. This synchronization can be achieved by sharing a common variable [31,32]. In previous studies, we have confirmed that by transmitting a signal from a dynamical system to its learned reservoir computer, synchronization will emerge [32]. Conversely, we will show that it is still possible to maintain synchronization even when the trained reservoir computer is used as the driving system. In particular, we find that by sending the x signal for driving, the y-variable of the Hénon map rapidly approaches to that of the trained reservoir system, see Fig. 3(a). This is further supported by observing the difference $\Delta y = \left|y' - y\right|$, which converges toward zeros, as shown in Fig. 3(b). Note that y' and y are generated from the reservoir computing system and the Hénon map, respectively. In addition, the same phenomenon also emerges on the Lorenz system for which the y component of the reservoir system is adopted as the driving signal (see Fig. 3(c) and (d)). These interesting synchronization phenomena can be referred to the sub-Lyapunov exponents, where there are the negative values [31,33]. Collectively, these findings point out that by sending a common signal, a chaotic system can synchronize with its learned reservoir computer and vice verse. This indirectly demonstrates that the trained reservoir computer can successfully model a chaotic system as an alternative to its dynamical equations.

Finally, to address the ability of reservoir computing approaches to model chaotic systems, we show that synchronization can still emerge in their coupled system. Here, we take the previous Hénon map as an example and use the x-variable

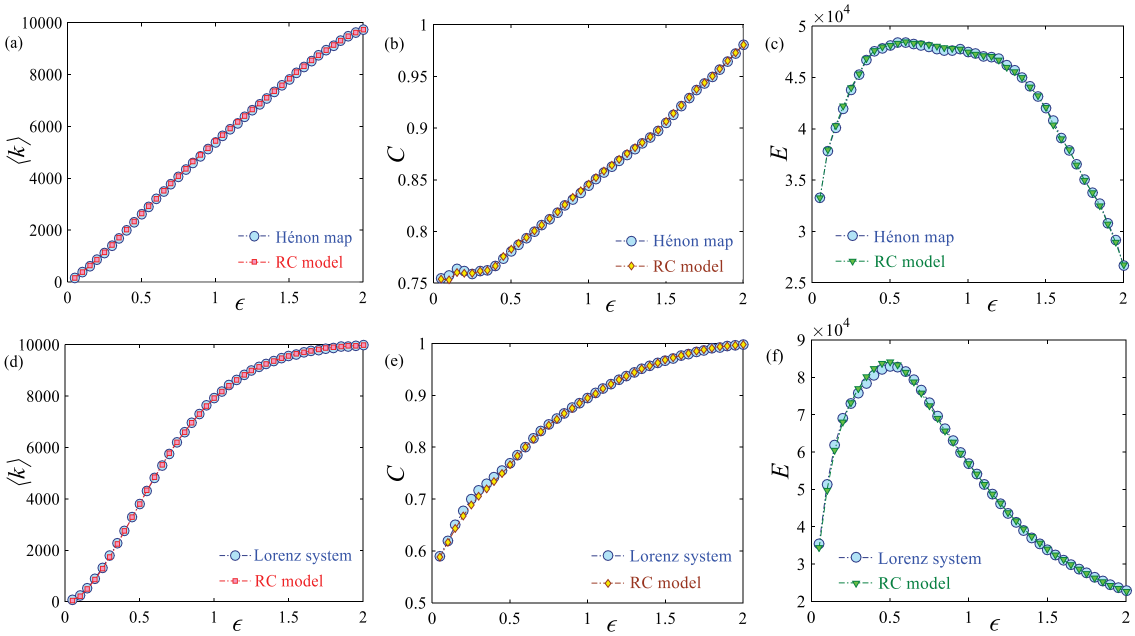

Fig. 2. Comparison of the Hénon map and its learned reservoir computer with respect to different network statistics: (a) the mean degree $\langle k\rangle$, (b) the clustering coefficient C, and the energy E. (d–f) The same comparison taking place on the Lorenz system.

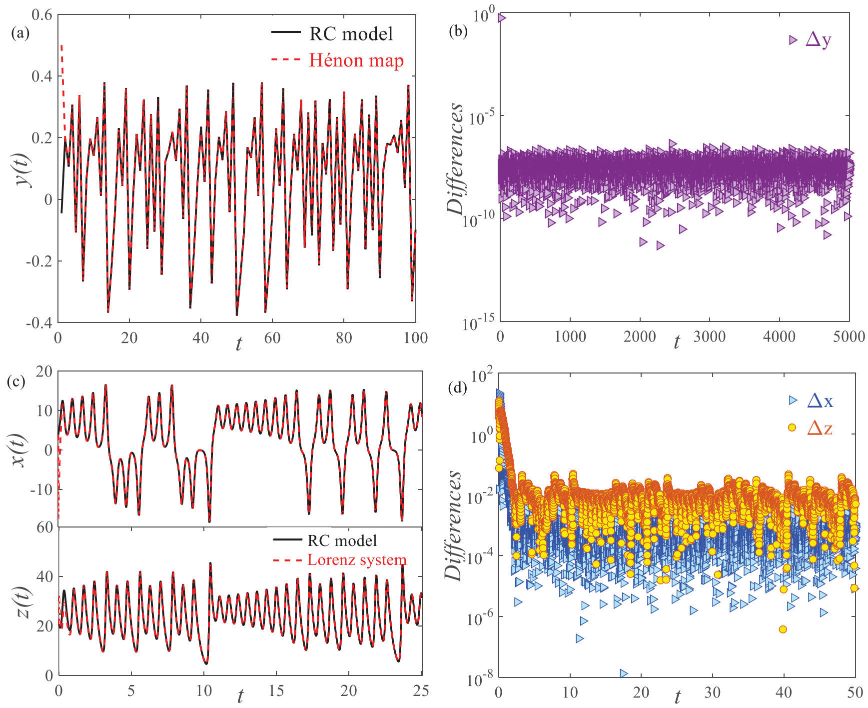

Fig. 3. (a) Complete synchronization of the trained reservoir computer and the Hénon map in terms of the y variable. (b) The associated difference Δy in function of time for the Hénon map. (c) Synchronization of the Lorenz system and the trained reservoir computer with the y-driven configuration. (d) The corresponding differences Δx and Δz between the trained reservoir computer and the Lorenz system.

as the coupling component. Specifically, the coupled system is defined as follows:

$$
\begin{aligned}
x_n &= x_n + \rho(x'_n - x_n),\\
x'_n &= x'_n + \rho(x_n - x'_n),\\
x_{n+1} &= 1 + y_n - 1.4x_n^2,\\
y_{n+1} &= 0.3x_n,\\
\mathbf{r}(n+1) &= (1-\alpha)\mathbf{r}(n) + \alpha\tanh\left(\mathbf{A}\mathbf{r}(n) + \mathbf{W}_{in}\begin{pmatrix} b_{in}\\ x'_n\\ y'_n\end{pmatrix}\right),\\
\begin{pmatrix} x'_{n+1}\\ y'_{n+1}\end{pmatrix} &= \mathbf{W}_{out}\begin{pmatrix} b_{out}\\ x'_n\\ y'_n\\ \mathbf{r}(n+1)\end{pmatrix},
\end{aligned}
\tag{6}
$$

where ρ is the coupled strength lying in the interval [0,1] and $\alpha = 0.25$. Here, the reservoir parameters $\mathbf{A}$, $\mathbf{W}_{in}$, b_{in}, and b_{out} are given in advance before training [11–13]. $\mathbf{W}_{out}$ is predefined based on training as we done for learning the previous Hénon map. The initial values of x_1, y_1, x'_1, and y'_1 are chosen randomly.

Fig. 4(a) depicts the average differences $\sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - x'_i)^2 + (y_i - y'_i)^2}$ as a function of the coupled strength. The differences tends to zeros when ρ is larger than 0.2. Interestingly, we find before realizing the synchronization, the periodic regime occurs at the interval $\rho \in [0.102, 0.138]$ (see Fig. 4(b)). This is further supported by observing the largest Lyapunov exponent λ of the coupling system for which $\lambda < 0$. We also notice that with increasing coupling strength ρ, the largest Lyapunov exponent λ gradually decreases and then sharply increases to approach a constant value. This means that the coupled system transitions from chaotic regime to periodic one and then to chaotic regime again. Remarkably, we observe that in the synchronization regime, the mean recurrence time $\langle T\rangle$ presents a similar power law behavior regardless of the coupled strength as shown in Fig. 4(c). The associated scaling exponents d are almost keeping constant, see the inset in Fig. 4(c). Meanwhile, besides the mean recurrence time $\langle T\rangle$, we find that the variation σ of the recurrence times exhibits an almost identical profile in synchronization regime as illustrated in Fig. 4(d). These findings further support that the trained reservoir system can play the same role as the analytic equations in describing a chaotic system.

4. Conclusions

In summary, we re-examined reservoir computing approach for modeling chaotic systems. We find that reservoir computer model provides an alternative to dynamical equations for describing a chaotic system. Specifically, we show that recurrence properties of the reservoir system are the same as that of dynamical equations. This equivalence is further confirmed in complex network domain for which a great variety of network statistics are almost identical. Moreover, we show that synchronization can occur between a chaotic system and its learned reservoir computer as well as in their coupled system, which indirectly evidence that reservoir computer model plays the same role as analytical equations in characterizing chaotic systems. Our work reveals that reservoir computing approach provides a new path for modeling and characterizing chaotic systems, whose equations of motion are unknown.

CRediT authorship contribution statement

Tongfeng Weng: Designed the research, Performed the research, Writing – original draft. **Huijie Yang:** Designed the research, Performed the research, Writing – original draft. **Jie Zhang:** Designed the research, Performed the research, Writing – original draft. **Michael Small:** Designed the research, Performed the research, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (NNSFC) under Grant Nos. 11805128 (T. Weng), and 10975099 (H. Yang). T. Weng acknowledges a Xihu Scholar award from Hangzhou City and the Hangzhou Normal University Starting Fund (4135C50220204098).

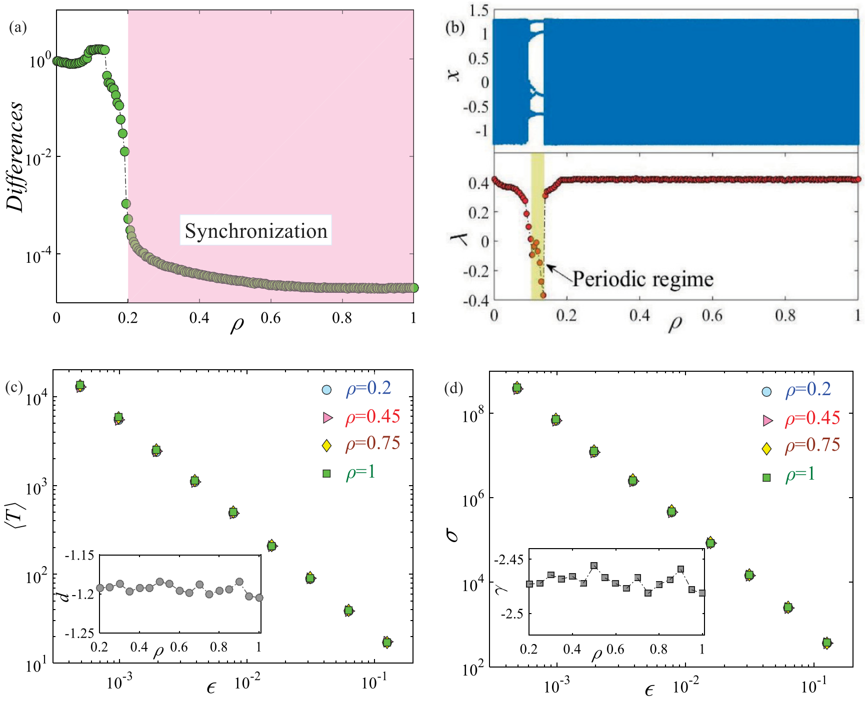

Fig. 4. (a) The differences vs coupling strength ρ. Synchronization occurs at the interval $\rho \in [0.2, 1]$. (b) The diagram of the x-variable and the maximum Lyapunov exponent λ as a function of the coupling strength ρ. (c) The mean recurrence time $\langle T \rangle$ as a function of ϵ over different coupling strength ρ. (d) The variation σ as a function of ϵ over different coupling strength ρ. In the insets, we show the profiles of the estimated values of d and γ vs ρ.

References

[1] Lorenz E. Deterministic nonperiodic flow. J Atmos Sci 1963;20:130–41.
[2] Li TY, Yorke JA. Period three implies chaos. Am Math Mon 1975;85:77–84.
[3] Kantz H, Schreiber T. Nonlinear time series analysis. Cambridge University Press; 2004.
[4] Marwan N, Romano MC, Thiel M, Kurths J. Recurrence plots for the analysis of complex systems. Phys Rep 2007;438:237–329.
[5] Hunt BR, Ott E. Defining chaos. Chaos 2015;25:097618.
[6] Zou Y, Donner RV, Marwan N, Donges JF, Kurths J. Complex network approaches to nonlinear time series analysis. Phys Rep 2019;787:1–97.
[7] Zhang J, Small M. Complex network from pseudoperiodic time series: Topology versus dynamics. Phys Rev Lett 2006;96:238701.
[8] Schmelcher P, Diakonos FK. Detecting unstable periodic orbits of chaotic dynamical systems. Phys Rev Lett 1997;78:4733.
[9] Jaeger H, Haas H. Harnessing nonlinearity: Predicting chaotic systems and saving energy in wireless communication. Science 2004;304:78–80.
[10] Manjunath G, Jaeger H. Echo state property linked to an input: Exploring a fundamental characteristic of recurrent neural networks. Neural Comput 2013;25:671.
[11] Pathak J, Hunt B, Girvan M, Lu ZX, Ott E. Model-free prediction of large spatiotemporally chaotic systems from data: A reservoir computing approach. Phys Rev Lett 2018;120:024102.
[12] Pathak J, Wikner A, Fussell R, Chandra S, Hunt B, Girvan M, et al. Hybrid forecasting of chaotic processes: Using machine learning in conjunction with a knowledge-based model. Chaos 2018;28:041101.
[13] Grigoryeva L, Ortega JP. Echo state networks are universal. Neural Netw 2018;108:495–508.
[14] Vlachas PR, Pathak J, Hunt BR, Sapsis TP, Girvan M, Ott E, et al. Backpropagation algorithms and reservoir computing in recurrent neural networks for the forecasting of complex spatiotemporal dyanmics. Neural Netw 2020;126:191–217.
[15] Fan HW, Jiang JJ, Zhang C, Wang XG, Lai YC. Long-term prediction of chaotic systems with machine learning. Phys Rev Res 2020;2. 012080(R).
[16] Lu ZX, Pathak J, Hunt BR, Girvan M, Brockett R, Ott E. Reservoir observers: Model-free inference of unmeasured variables in chaotic systems. Chaos 2017;27:041102.
[17] Nakai K, Saiki Y. Machine-learning inference of fluid variables from data using reservoir computing. Phys Rev E 2018;98:023111.
[18] Carroll TL. Using reservoir computers to distinguish chaotic signals. Phys Rev E 2018;98:052209.
[19] Pathak J, Lu ZX, Hunt BR, Girvan M, Ott E. Using machine learning to replicate chaotic attractors and calculate Lyapunov exponents from data. Chaos 2017;27:121102.
[20] Lu ZX, Hunt BR, Ott E. Attractor reconstruction by machine learning. Chaos 2018;28:061104.
[21] Haluszczynski A, Räth C. Good and bad predictions: Assessing and improving the replication of chaotic attractors by means of reservoir computing. Chaos 2019;29:103143.
[22] Gao JB. Recurrence time statistics for chaotic systems and their applications. Phys Rev Lett 1999;83:3178.
[23] Jiang J J, Lai YC. Model-free prediction of spatiotemporal dynamical systems with recurrent neural networks: Role of network spectral radius. Phys Rev Res 2019;1:033056.

[24] Lukoševicius M, Jaeger H. Reservoir computing approaches to recurrent neural network training. Comput Sci Rev 2009;3:127–49.
[25] Zimmermann RS, Parlitz U. Observing spatio-temporal dynamics of excitable media using reservoir computing. Chaos 2018;28:043118.
[26] Gao JB, Cai HQ. On the structures and quantification of recurrence plots. Phys Lett A 2000;270:75–87.
[27] Xu XK, Zhang J, Small M. Superfamily phenomena and motifs of networks induced from time series. Proc Natl Acad Sci USA 2008;105:19601–5.
[28] Donner RV, Zou Y, Donges JF, Marwan N, Kurths J. Recurrence networks-a novel paradigm for nonlinear time series analysis. New J Phys 2010;12:033025.
[29] Donner RV, Small M, Donges JF, Marwan N, Zou Y, Xiang RX, et al. Recurrence-based time series analysis by means of complex network methods. Int J Bifurcation Chaos 2011;21:1019–46.
[30] Gutman I, Zhou B. Laplacian energy of a graph. Linear Algebra Appl 2006;414:29–37.
[31] Pecora LM, Carroll TL. Synchronization in chaotic systems. Phys Rev Lett 1990;64:821.
[32] Weng TF, Yang HJ, Gu CG, Zhang J, Small M. Synchronization of chaotic systems and their machine-learning models. Phys Rev E 2019;99:042203.
[33] Boccaletti S, Kurths J, Osipov G, Valladares DL, Zhou CS. The synchronization of chaotic systems. Phys Rep 2002;366:1–101.